The Paradox of the Two Envelopes

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1. The Paradox of the Two Envelopes

Initial Situation. There are two envelopes in front of you. You know nothing about what they contain, except that one contains an amount of money twice as large as the other. You pick an envelope; call it M for "mine", and call the other one O, for "other". You are told that you will earn whatever amount is in the envelope you have, and you are given the opportunity to change envelopes. What should you do?

Paradoxical Reasoning. Let x be the amount in M. You consider two possibilities: either O contains $x/2$, or it contains $2x$. They are epistemically equally likely. You represent this decision problem in the following matrix **1**.

	O contains $x/2$	O contains $2x$
Stick	x	x
Change	$x/2$	$2x$

Matrix 1

The expected utilities of sticking and changing are, respectively:

$$E[S] = x/2 + x/2 = x$$

$$E[C] = 1/2(x/2) + 1/2(2x) = 5x/4$$

Supposing that you are an expected utility maximiser, it appears that it is rational for you to change to the other envelope. But:

(i) Given that both envelopes are assumed to be indistinguishable, how can one be rationally preferred to the other?

(ii) The same argument can be given for changing back, after having changed once, and so *ad infinitum*.

(iii) The same argument can be given for rationality of sticking, starting with the amount in O.

So, something must have gone wrong in the above argument. But what? This is the **two-envelope paradox**.

2. Outline of my Proposed Solution

General dialectic: there is a condition **(C)** which is implicitly assumed in the initial condition and which is violated (in an interesting way) in matrix **1**. So, matrix **1** cannot adequately represent the initial situation.

Roughly, **(C)** is the condition that for any amount of money, if you think it's possible for that amount to be found in M, then you also think that it's possible for it to be found in O. For instance, if you think it's possible for there to be £4 in your envelope, then you also think it's possible for there to be £4 in the other envelope.

So, if there is a column where you get some amount in some row, there is another column where you get that same amount in the other row. Look at matrix **1** again: it visibly violates **(C)**.

3. Some Reasons to Keep Listening

You might be worried that there is something fishy with using "x" and I want to put this worry somewhat to rest: there are non-paradoxical matrices that do not use numerals, and there are paradoxical matrices that do.

	M contains x O contains $x/2$	M contains x O contains $2x$
Stick	x	x
Change	$x/2$	$2x$

Matrix 1

	M contains x O contains $2x$	M contains $2x$ O contains x
Stick	x	$2x$
Change	$2x$	x

Matrix 2

	w_1	w_2	w_3	w_4
S	2	2	4	4
C	1	4	2	8

Matrix 3

	w_0	w_1	w_2	w_3	w_4	w_5
S	1	2	2	4	4	8
C	2	1	4	2	8	4

Matrix 4

The difference between matrices **1** and **2** on the one hand, and **3** and **4** on the other, is the same: matrices **1** and **3** violate condition **(C)**, whereas matrices **2** and **4** do not. What we need to change in **3** to get **4**, is to add the possible states w_0 and w_5 to our decision matrix. But, those are precisely the ones that would fulfil condition **(C)**, and also precisely the ones that we can't add to **3** if it's going to be a fine-graining of **1**.

4. A Puzzle about Designation

I want to explain what is wrong with matrix **1**. So far, I have established (I hope) that matrix **1** violates condition **(C)**. So, I must do two things: (§4) explain *why* matrix **1** violates condition **(C)**, and (§5-7) explain why condition **(C)** must be respected if we hope to represent the initial situation appropriately.

Contention. Matrix **1** violates condition **(C)** because of what "x" designates in that matrix, namely the amount in M. Indeed, if "x" designates whatever amount M contains, and if O must contain twice or half as much as M, then x cannot be in O. This violates **(C)**, which states that, for any amount, if the agent considers it possible that this amount is in one envelope, she also must consider it possible that this amount is in the other envelope.

Contention. The paradox of the two envelopes is, at heart, a problem about designation; namely the problem of explaining *why* using "x" to designate whatever is in M leads us astray. (Note: using "x" to designate whatever is in M cannot be bad in general; there doesn't seem to be any problems with saying: "If the amount in M is greater than £50, I'm buying everyone drinks!")

5. Rigid Designators in State Descriptions

There are arguments in the literature that the state descriptions must include rigid designators only. If these arguments are successful, this is good news: it explains why matrix **1** is paradoxical.

> **Katz and Olin** (2007): "if we were not using "[x]" rigidly, we could not make sense of the comparisons of utility across different possible states of the world" (p. 909).
>
> **Horgan** (2000): "the argument [is] therefore bogus, since "x" lacks a single constant referent" (p. 584).

A counterexample. I have invited my sister, Agatha, and my friend, Camilla, over for tea. I leave them for a second to pour the water into the teapot and, when I come back, my chocolate bar is gone. I have the choice between making a remark and remaining silent. If Agatha took the chocolate bar and I make a remark, she will give me one of hers and I will have chocolate after all. Furthermore, I will feel no embarrassment. However, if Camilla took the chocolate, she would also give me one of her chocolates, but I would be mortified for having said something. I represent the problem in the following decision matrix:

	Agatha is the culprit	Camilla is the culprit
Say something	Get chocolate	Get chocolate and be mortified
Remain silent	Get no chocolate	Get no chocolate

Matrix 5

In this case, it seems quite clear that I could (at least in principle) assign utilites to the possible outcomes, probabilities to the possible states, and make an expected utility calculation. But "the culprit" is a non-rigid designator: it refers to Agatha in one state, and to Camilla in the other. Now you might think that, all that matters is that I could have written the states with rigid designators only ("Agatha took my chocolate", "Camilla took my chocolate"). But then consider a non-identity case:

	My child will have a 14 year old parent	My child will have a 28 year old parent
Have child now	a	b
Have child later	c	d

Matrix 6

So, actually, there doesn't seem to be any issues with non-rigid state descriptions. Maybe we can't explain the paradoxicality of **1** by appealing to rigidity concerns.

	The amount in M is the smallest	The amount in M is the largest
Stick	The smallest amount	The largest amount
Change	The largest amount	The smallest amount

Matrix 7

	M contains $x/3$ O contains $2x/3$	M contains $2x/3$ O contains $x/3$
Stick	$x/3$	$2x/3$
Change	$2x/3$	$x/3$

Matrix 8

This transfers to the two envelope paradox. Matrices **7** and **8** both model the initial situation. However, "the amount in M" refers to two different entities in the two different states of matrix **7**, and "x" refers to the same entity in the two different states of matrix **8**. That is, the states have been picked out with a non-rigid designator in one case, and with a rigid one in the other. So, using non-rigid designators across states to pick out those states does not automatically render the expected utility calculations "bogus" or "nonsensical", even in the case of the two-envelope paradox.

6. Transparent Designators

State descriptions, credences and utility assignments are bound to be intensional. Horgan (2000) argues that, in decision theory, we should only use "canonical" or "tranparent" designators. But this does not solve the paradox of the two envelopes. Indeed, canonicity and paradoxicality are indepdent (figure 1). So, the former cannot explain the latter.[1]

	Canonical	Non-Canonical
Paradoxical	Matrix **3**	Matrix **1**
Non-Paradoxical	Matrix **2**	Matrix **9**

Figure 1

	O contains $1/2$(Heidi)	O contains 2(Lois)
Stick	Heidi	Lois
Change	$1/2$(Heidi)	2(Lois)

Matrix 9

In any case, the notion of transparency or canonicty is at best unclear. When is a description canonical? When it uses rigid designators designators?[1] When we (can) know what it describes? When we can recognise that two transparent descriptions describe the same thing?

7. Rigidity and Utility Calculations

The coin flip situation. You are given an envelope, call it M, containing some amount of money x. Opposite you is another envelope, O. A fair coin is flipped and, if it lands heads, $2x$ is placed in the other envelope, and if it lands tails, $1/2$ is placed in O. In this case, it is obviously rational to switch to the other envelope. Indeed, the expected utility of changing is higher than that of sticking for any x.

Reformulation of the two-envelope paradox. What is the salient difference between the initial situation of the two-envelope paradox, and the coin flip situation? Why is it fine to use x to denote the amount in M in the latter, but not the former?

Contention. The salient distinction between the two cases has nothing to do with causation, counterfactuals or temporal ordering. Rather, it has to do with what may evasively be called "point of view", and which is pinned down precisely by considerations of rigidity in the utility calculations. In the coin flip case, the utility you get from keeping M is indeed the same across states, but not in the initial situation.

If we use x to denote the amount in M, and we use x in utility calculations, then x must be a rigid designator. But the initial situation is such that x should not refer rigidly. This is why using x to designate whatever is in M leads us astray. This explains the violation of **(C)**: if x rigidly refers to the amount in M, there is some amount that can be in M but not in O, namely x.

8. Conclusion

We have two distinct decision problems. One is the initial situation, for which it is not rational to change, and which is not adequately described by **1**. The other is the coin flip situation, for which it is rational to change, and which is described by **1**. The difference is that the rigidity of x is required by the utility calculation, and that this assumption holds only in the coin flip case, and not in the initial situation.

References

Horgan, T. (2000). The Two-Envelope Paradox, Nonstandard Expected Utility, and the Intensionality of Probability. *Noûs 34*(4), 578–603.

Katz, B. D. and D. Olin (2007). A Tale of Two Envelopes. *Mind 116*(464), 903–926.

[1] Note that canonicity and rigidity are also independent.